

A homage to my departed Guru, Professor Sudhansu Datta Majumdar

Debabrata Basu

In winter, when the fields are white

I sing this song for your delight

In spring when the woods are getting green

I shall try and tell you what I mean

We were the last batch of M.Sc. (Pure Physics) students of Calcutta University taught by Professor Sudhansu Datta Majumdar. We appeared for final examination in the winter of 1965 when Prof. Datta Majumdar shifted to Indian Institute of Technology Kharagpur. Although most of his contemporaries at the Saha Institute of Nuclear Physics, Indian Association for the Cultivation of Science and similar citadels of elitism in Kolkata used to speak of him with derision I had the common sense to ignore those remarks and prepare myself for future work with him. I was only too keenly aware of my own limitations and felt that a little more maturity would be required to cope with his problems. It is precisely for this reason that I tried to secure a research scholarship under Professor S. N. Biswas of Delhi University who would be more suited to an immature intellect like me. I hoped at least to get a post doctoral scholarship or some such position at I. I. T. Kharagpur. So, when Professor H. N. Bose offered me a job it was an unexpected windfall (for me). I wrote to Prof. SDM asking him whether he would accept me in his group and he promptly responded by welcoming me to work with him.

At first he advised me to continue my work in high energy physics. When I declined and expressed my desire to work on group representation theory he asked me to see whether something can possibly be done on the representations of locally compact groups. As my background on representation theory was absolute zero I was advised to start from the works of the Russian experts whose artless simplicity appealed to his natural sympathy. I little understood at that time that the monumental volumes he handed me down were destined to be my lifelong companions and the quest he set me on would continue for the remainder of my natural existence. Although some of my intimate friends regarded it as a futile pursuit of elitism a deeper man called it a variant of worship.

Prof. SDM admitted that he knew nothing of the subject. It was, therefore, decided that every week I should give two to three talks in his room, he himself being the sole audience. I had to start from the scratch, namely, $SU(2)$. At that time, I was unaware of his famous work on the Clebsch-Gordan problem of $SU(2)$ which would be the starting point of all my subsequent works with him. At first I started learning the basics: the Wigner–Racah method of construction of the representations and its reduction into irreducible parts using the spinor representation of the Lie algebra etc. He knew by heart this form of the representation theory. SDM became much more alert when I started speaking about the Lorentz group or $SL(2, \mathbb{C})$. Ultimately when I had described the Gel’fand–Naimark construction of the principal series of representations of $SL(2, \mathbb{C})$ he asked me to pause awhile. He appeared stunned by the discovery and delivered himself of his memorable judgement: *This is an epoch-making discovery of this century.* When I had completed their construction of the supplementary series of representations his admiration turned ecstatic. His judgement regarding the novelty of the representation theory to the effect that it needs no basis, no infinitesimal operators i.e. differentiability at the identity, even no separate coordinate space, was a revelation to a novice on the threshold of this subject. SDM’s admiration of the theory based on the finite element of the group acting as operators on the coset spaces, which unravels the whole gamut of unitary representations like a miracle, seemed to surpass all those of their predecessors.

SDM pointed out that we are too unequal to the experts in our mathematical acumen and consequently it would be better to start the subject *in our own way* which is certain to reproduce many known results but at the same time help us, at least, to understand it better. At first, we started by reducing $SL(2, \mathbb{C})$ under its maximum compact $SU(2)$ subgroup. The bases for this reduction turned out to be an analytic continuation of SDM’s well known hypergeometric function which appears in the Clebsch–Gordan problem of $SU(2)$. This yields in addition to all well-known results of Gel’fand and co-workers and Harish–Chandra the matrices of finite transformation of $SL(2, \mathbb{C})$ (in the $SU(2)$ basis). The investigation of the $SU(1,1)$ content of $SL(2, \mathbb{C})$ led to different analytic continuation of the same hypergeometric function both in parameters and in variables. The next problem was the Clebsch–Gordan problem of $SU(1,1)$. We could complete only the Kronecker products $D^+ \times D^\pm$, $D^+ \times D^C$, $D^C \times D^C$. In the last product Majumdar’s hypergeometric function had to be defined on the unit circle and consequently a Fourier expansion of the hypergeometric function was needed

for the evaluation of the Clebsch-Gordan coefficients. Here we ran into an unexpected difficulty. Evaluation of the Fourier coefficients led to an integral which for $m < 0$ seemed to diverge. With a look of extreme chagrin SDM gave vent to his frustration: *What are you going to do? what bloody fools we are!* That is the only occasion in which I have noticed a sanguinary element in his language.

This was the time of his superannuation and it was decided that I shall meet him in Kolkata after he left Kharagpur. I went to Kolkata during the summer vacation and after beating around the bush for more than a week I could settle the point to my satisfaction. I was so happy with the beauty of the solution that next morning I rushed to his residence in Jadavpur from my home in Paikpara and showed him the solution. With a broad grin the old man then showed me his own resolution. The two were practically identical.

Even at the risk of making this write-up too technical I am giving below the bare skeleton of the problem and its resolution because it forms, in essence, the key to several chapters of Gel'fand and Shilov's classic treatise *Generalised functions (Volume 1)*. Let us start from an almost *trivial* example.

$$\frac{1}{z} = \int_0^1 t^{z-1} dt, \text{ Re } z > 0 \quad (1)$$

Although for $\text{Re } z < 0$ (left half-plane) the integral diverges, it is natural to enquire whether it is possible to modify the *integral representation* such that it remains meaningful outside the region of convergence of the line integral and reduces to it for $\text{Re } z > 0$. This is precisely what is achieved in Cauchy's theory of analytic continuation by replacing the line integral as an integral over the contour C which starts from $t = 1$, encircles the origin ($t = 0$) once counterclockwise and returns to $t = 1$. One can then write

$$\frac{1}{z} = \frac{\mu(z)}{e^{2\pi iz} - 1}; \quad \mu(z) = \int_C t^{z-1} dt \quad (2)$$

An alternative notation would be

$$\mu(z) = \int_1^{0+} t^{z-1} dt \quad (3)$$

Proof: It should be noted that on upper linear part of Fig 1. $\arg t = 0$ while on lower linear part $\arg t = 2\pi$. Thus

$$\mu(z) = (e^{2\pi iz} - 1) \int_{\epsilon}^1 t^{z-1} dt + \int_s t^{z-1} dt \quad (4)$$

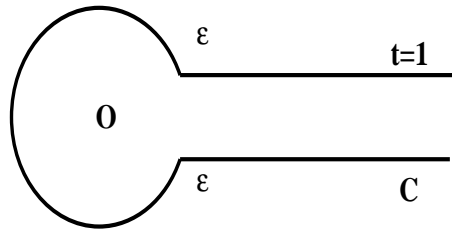


FIG. 1: The contour C

where s stands for the circle of radius ϵ . Hence

$$\begin{aligned}
 \mu(z) &= (e^{2\pi iz} - 1) \frac{(1 - \epsilon^z)}{z} + i\epsilon^z \int_0^{2\pi} e^{i\theta z} d\theta \\
 &= (e^{2\pi iz} - 1) \frac{(1 - \epsilon^z)}{z} + \frac{\epsilon^z}{z} (e^{2\pi iz} - 1) \\
 &= \frac{(e^{2\pi iz} - 1)}{z}
 \end{aligned} \tag{5}$$

Thus

$$\frac{1}{z} = \frac{\mu(z)}{(e^{2\pi iz} - 1)} = \frac{1}{(e^{2\pi iz} - 1)} \int_1^{0+} t^{z-1} dt \tag{6}$$

is finite for all z ($z \neq 0$) and the integral is independent of the radius ϵ of the circle.

The role of the denominator $(e^{2\pi iz} - 1)$ appearing outside the integral is misunderstood by the authors of several standard books of classical analysis. For example these authors state that the contour integral representation of the Γ function

$$\Gamma(z) = \frac{1}{(e^{2\pi iz} - 1)} \int_{\infty}^{0+} e^{-t} t^{z-1} dt \tag{7}$$

breaks down at $z = 0, \pm 1, \pm 2, \dots$

But actually it is regular at $z = 1, 2, 3, \dots$ (unless, of course, one considers z/z singular at $z = 0!$) since it can be easily verified that,

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + E(z) \tag{8}$$

where $E(z)$ is an entire function.

Once this part was understood the rest of the problem was simple. For representations belonging to the principal series Majumdar's hypergeometric equation has two linearly independent solutions which were chosen such that they were orthogonal to each other. The representation space, therefore, splits into two orthogonal subspaces leading to double multiplicity in the Clebsch–Gordan series. Evaluation of the normalisation constant was carried

out by the traditional method using the Sturm–Liouville theory of second order differential equation leading to a pair of orthonormal Clebsch–Gordan coefficients (in the $SO(2)$ basis).

I rarely met SDM after he joined the Mathematics Department of Visva Bharati University. Before we parted he told me there are two ways ahead of me: (1) First, whatever we have done so far are amateurish works on harmonic analysis. For survival in this subject I should do something *professional*. When I asked him what a professional work would look like he pointed out that it does not have to be at par with problems treated by the Russian group or by Prof. Harish–Chandra which only a gifted few can do. But it has to deal with the basic invariants of the group. For a person with no formal training in mathematics taking this path would be perilous treading on the razor’s edge which may turn out to be rather bloody. (2) Second, I may come back to theoretical physics (in which I never had any confidence) and try to apply the experience gained so far to physical problems.

During the first two or three years I suffered from an acute feeling of insecurity to lose the support of such a towering intellect beside me. But gradually self–confidence, which, in this instance might have been an euphemism for obstinacy contracted from my proximity to two typical representatives of East Bengalee Obstinacy (SNB and SDM) started creeping back to my system.

My first really independent work in this direction dealt with the finite element of $SL(2, \mathbb{R})$ for a special representation in a Hilbert space of analytic functions which led to a class of integral transform pairs. Here I first learned the concept and use of principal vectors.

Next I attempted a more serious implementation of SDM’s suggestion by making an in–depth study of the concept of character introduced by the Russian group. This seemed to me a major step ahead of the traditional concept which not only makes a complete synthesis of the finite and infinite dimensional irreducible unitary representations but introduces the concept in an invariant way by using the integral kernel of the group ring. I completed the recalculation of the character of all representations not only of $SL(2, \mathbb{R})$ but of its universal cover with the able collaboration of two of my bright students of I. I. T. Kharagpur. But the derivation of the Plancherel formula, which is essentially the statement of Peter–Weyl theorem remained unsolved. Ultimately the hint came from Vilenkin and Klimyk’s magnum opus on *Representations of Lie Groups*. The problem was completely solved (arxiv:0710.2224v1, hep-th) more than two decades after SDM had sowed its seeds.

My friend GPS in his inimitable homage has compared SDM with Richard Feynman.

But in my opinion his personality was very different from a playboy Feynman or a saintly Gel'fand. He was nearer Hermann Weyl in temperament and taste if not in gift. To recognise him as such would be the most befitting homage to his memory.

Good men are seldom given a place in the galaxy of the great. It will continue to be so until goodness is recognised as a true measure of greatness. He was kind and truthful as well as bold and steadfast. West Bengal will not see the like of him again.

Note: The initials used in the article stand for:

SDM → Professor S. Datta Majumdar

SNB → Professor S. N. Biswas

GPS → Professor G. P. Sastry

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